



Comparative Analysis of Analytical and Numerical Methods for Nonlinear Fractional-Order Differential Equations

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ABSTRACT

Nonlinear fractional-order differential equations constitute an important area of mathematical analysis because they incorporate nonlocal operators and memory-dependent behaviour. This study compares analytical and numerical techniques for nonlinear fractional differential equations defined using the Caputo derivative. The equivalent Volterra integral formulation supports two algorithms: an explicit fractional product-integration scheme and an Adams-type predictor–corrector method. Two nonlinear benchmarks are studied: a manufactured-solution equation allowing direct exact-error computation, and a fractional Riccati equation evaluated using a fine-grid numerical reference. Tests at fractional orders $\alpha = 0.5$ and $\alpha = 0.8$ on successively refined grids are assessed using maximum absolute error, solution profiles and empirical convergence orders. Across these examples, product integration exhibits approximately first-order convergence, whereas the predictor–corrector method exhibits higher convergence orders broadly consistent with $1 + \alpha$. The distinction between exact error and reference-based error is emphasized, and the benchmark-specific nature of the numerical findings is explained.

Keywords: nonlinear fractional differential equations; Caputo derivative; fractional calculus; product integration; predictor–corrector method; Riccati equation; convergence order; numerical error.

1. INTRODUCTION

Fractional calculus generalizes differentiation and integration to non-integer orders and provides a rigorous framework for nonlocal memory-dependent systems. In the Caputo formulation, the derivative at time t depends on the history of the function over an earlier time interval. This makes fractional models mathematically richer than their ordinary differential equation analogues, but also makes their exact and approximate solution more demanding.

Nonlinearity compounds these challenges. Fixed-point methods, integral formulations and successive approximation can provide existence, uniqueness or analytical representations for particular problems, but closed-form expressions are often unavailable. Numerical methods based on the Volterra formulation offer a practical alternative by approximating the memory integral at discrete times.

Diethelm, Ford and Freed (2002) developed an influential Adams-type predictor–corrector approach to fractional initial-value problems. Lubich (1986) examined discretized fractional calculus, and Garrappa (2018) reviewed numerical approaches and practical implementation issues. A meaningful comparison needs not only solution values but also verified errors, grid-refinement rates and clarity about whether the reference is exact or numerical.

1.1 Objectives

The objectives are to formulate a Caputo nonlinear initial-value problem; verify an analytical benchmark; implement explicit product integration and an Adams-type predictor–corrector scheme; measure absolute errors and empirical convergence orders; examine a fractional Riccati equation without a prescribed exact solution; and assess sensitivity to the fractional order and grid size.

2. MATHEMATICAL FOUNDATIONS

2.1 Fractional integral and derivative

For an integrable function f and order $\alpha > 0$, the Riemann–Liouville fractional integral is

$$I^\alpha f(t) = [1/\Gamma(\alpha)] \int_0^t (t-s)^{\alpha-1} f(s) ds. \quad (1)$$

The gamma function satisfies $\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du$ for $\text{Re } z > 0$. For $\beta > -1$,

$$I^\alpha t^\beta = [\Gamma(\beta+1)/\Gamma(\alpha+\beta+1)] t^{\alpha+\beta}. \quad (2)$$

For $0 < \alpha < 1$ and sufficiently regular y , the Caputo fractional derivative is

$${}_CD^\alpha y(t) = [1/\Gamma(1-\alpha)] \int_0^t (t-s)^{-\alpha} y'(s) ds. \quad (3)$$

$${}_CD^\alpha t^\beta = [\Gamma(\beta+1)/\Gamma(\beta+1-\alpha)] t^{\beta-\alpha} \quad (\beta > 0). \quad (4)$$

2.2 Nonlinear initial-value problem

$${}_CD^\alpha y(t) = f(t, y(t)), y(0) = y_0, 0 < \alpha < 1. \quad (5)$$

Applying the fractional integral gives an equivalent nonlinear Volterra equation:

$$y(t) = y_0 + [1/\Gamma(\alpha)] \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds. \quad (6)$$



If f is continuous and Lipschitz in its second argument on a suitable domain, the associated operator admits a contraction argument over an appropriately restricted interval. A sufficient contraction condition in the uniform norm is $LT^\alpha/\Gamma(\alpha + 1) < 1$; this condition is not necessary for uniqueness.

3. ANALYTICAL SOLUTION FRAMEWORK

3.1 Successive approximation

The integral form yields a fractional Picard iteration, initialized with a suitable $y^{(0)}$: $y^{(k+1)}(t) = y_0 +$

$$I^\alpha \left[f(t, y^{(k)}(t)) \right]. \quad (7)$$

Convergence depends on regularity, the Lipschitz condition and the solution interval. This analytical construction is particularly useful for studying solvability even when an elementary closed form is unavailable.

3.2 Exact-solution nonlinear benchmark

$${}_CD^\alpha y(t) = 2t^{2-\alpha}/\Gamma(3-\alpha) - y(t)^2 + t^4, y(0) = 0. \quad (8)$$

On $0 \leq t \leq 1$, direct substitution of $y(t) = t^2$ gives ${}_CD^\alpha t^2 = 2t^{2-\alpha}/\Gamma(3-\alpha)$; the nonlinear terms $-t^4 + t^4$ cancel, and the initial condition is satisfied. Thus the exact reference is $y(t) = t^2$ for every $0 < \alpha < 1$. This deliberately manufactured benchmark provides an independent basis for exact-error measurement.

3.3 Fractional Riccati benchmark

$${}_CD^\alpha y(t) = 1 - y(t)^2, y(0) = 0, 0 \leq t \leq 1. \quad (9)$$

The equivalent Volterra equation is $y(t) = I^\alpha[1 - y(t)^2]$. The classical $\alpha = 1$ equation has the solution $\tanh(t)$, but that expression is not used as an exact reference for $\alpha = 0.5$ or 0.8 . Instead, a predictor–corrector solution on $N_{ref} = 2560$ steps supplies a fine-grid numerical reference. Its finite error must be distinguished from exact-solution error.

4. NUMERICAL METHODOLOGY

Let $t_n = nh$, $h = 1/N$, for $n = 0, \dots, N$, and let Y_n approximate $y(t_n)$. The experiments use $N = 40, 80, 160$ and 320 at $\alpha = 0.5$ and 0.8 .

4.1 Explicit fractional product integration

$$Y_{n+1} = y_0 + [h^\alpha/\Gamma(\alpha + 1)] \sum_{j=0}^n b_{j,n+1} f(t_j, Y_j). \quad (10)$$

$$b_{j,n+1} = (n+1-j)^\alpha - (n-j)^\alpha. \quad (11)$$

This scheme approximates the integrand piecewise constantly and uses only available historical values.

4.2 Fractional Adams-type predictor–corrector

$$Y_{n+1}^P = y_0 + [h^\alpha/\Gamma(\alpha + 1)] \sum_{j=0}^n b_{j,n+1} f(t_j, Y_j). \quad (12)$$

$$Y_{n+1} = y_0 + [h^\alpha/\Gamma(\alpha + 2)] f(t_{n+1}, Y_{n+1}^P) + \sum_{j=0}^n a_{j,n+1} f(t_j, Y_j). \quad (13)$$

$$a_{0,n+1} = n^{\alpha+1} - (n-\alpha)(n+1)^\alpha. \quad (14)$$

$$a_{j,n+1} = (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, 1 \leq j \leq n. \quad (15)$$

One correction step is taken per grid point. Both direct algorithms retain memory of previous function values and have $O(N^2)$ arithmetic cost in straightforward implementations.

4.3 Accuracy and convergence measures

$$E_{\max} = \max_{0 \leq n \leq N} |y(t_n) - Y_n|. \quad (16)$$

$$E_{ref} = \max_{0 \leq n \leq N} |Y_n - Y_{ref}(t_n)|. \quad (17)$$

$$p_{obs} = \ln(E_h/E_{h/2})/\ln 2. \quad (18)$$

$E_{\max}$ applies to the manufactured benchmark with the known exact solution; E_{ref} applies to the Riccati problem. The reference-grid size is divisible by every tested grid size, so the comparison occurs at matching nodes.

5. COMPUTATIONAL RESULTS AND ANALYSIS

5.1 Exact-error benchmark results

Table 1. Maximum absolute errors for the analytical benchmark

α	N	Product integration	Predictor–corrector
0.5	40	0.01338311	0.00439572
0.5	80	0.00647540	0.00140655
0.5	160	0.00316792	0.00046237
0.5	320	0.00156071	0.00015520
0.8	40	0.01467687	0.00111702
0.8	80	0.00723944	0.00031167
0.8	160	0.00359329	0.00008785
0.8	320	0.00178964	0.00002492



Both schemes show decreasing maximum absolute errors as N increases. Predictor–corrector approximations have smaller errors throughout the tested configurations. For $\alpha = 0.8$ at $N = 320$, the predictor–corrector error is approximately 2.49×10^{-5} , compared with approximately 1.79×10^{-3} for product integration.

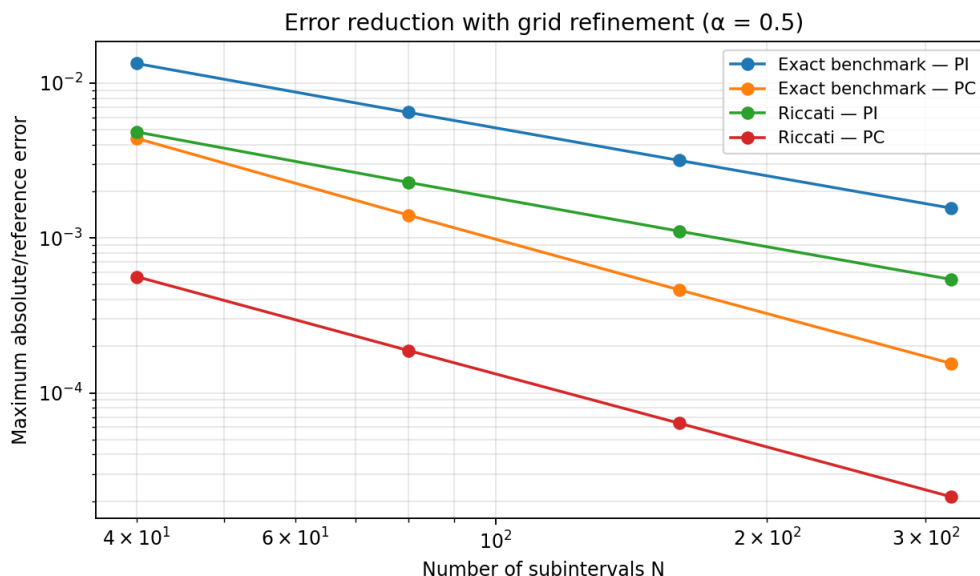


Figure 1. Error convergence at $\alpha = 0.5$ on logarithmic axes.

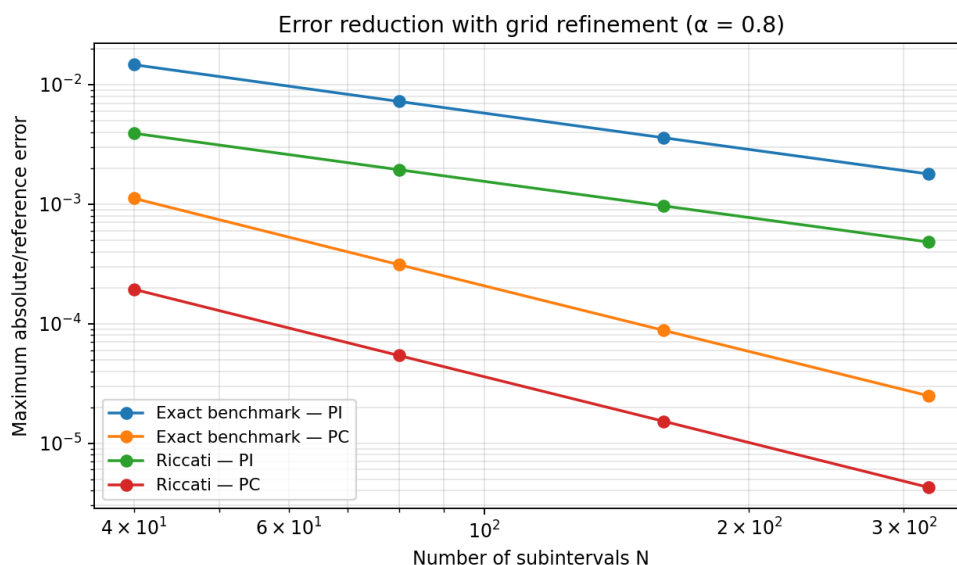


Figure 2. Error convergence at $\alpha = 0.8$ on logarithmic axes.

5.2 Explicit convergence-order calculations

For $\alpha = 0.5$, the product-integration rate from $N = 40$ to 80 is $\text{pobs} = \ln(0.01338311 / 0.00647540) / \ln(2) \approx 1.04737$. For predictor–corrector on the same grids, $\text{pobs} = \ln(0.00439572 / 0.00140655) / \ln(2) \approx 1.64394$.

Table 2. Empirical convergence orders for the analytical benchmark

α	Refinement	Product integration	Predictor–corrector
0.5	40 $\rightarrow$ 80	1.04737	1.64394
0.5	80 $\rightarrow$ 160	1.03143	1.60503
0.5	160 $\rightarrow$ 320	1.02134	1.57496
0.8	40 $\rightarrow$ 80	1.01959	1.84156
0.8	80 $\rightarrow$ 160	1.01057	1.82698
0.8	160 $\rightarrow$ 320	1.00563	1.81771

Product-integration estimates approach first-order convergence, while predictor–corrector estimates approach $1 + \alpha$ (1.5 or 1.8). Finite-grid orders can differ from the asymptotic theoretical order.



5.3 Fractional Riccati numerical reference

Table 3. Maximum reference-based errors for the fractional Riccati equation

α	N	Product integration	Predictor–corrector
0.5	40	0.00483766	0.00056288
0.5	80	0.00228885	0.00018810
0.5	160	0.00110588	0.00006365
0.5	320	0.00054096	0.00002138
0.8	40	0.00391315	0.00019360
0.8	80	0.00194161	0.00005410
0.8	160	0.00096685	0.00001524
0.8	320	0.00048239	0.00000427

Table 4. Numerical Riccati solution values at $t = 1$

α	N	Product integration	Predictor–corrector
0.5	40	0.700629	0.698325
0.5	80	0.699679	0.698602
0.5	160	0.699207	0.698693
0.5	320	0.698973	0.698723
0.8	40	0.747058	0.743264
0.8	80	0.745249	0.743402
0.8	160	0.744350	0.743441
0.8	320	0.743903	0.743452

The fine-grid reference values at $t = 1$ were approximately 0.698739 ($\alpha = 0.5$) and 0.743456 ($\alpha = 0.8$). The tested approximations approach these values as the grid is refined. Because the reference is numerical, agreement cannot be interpreted as an independently established exact error.

Fractional Riccati solution comparison ($N = 40$)

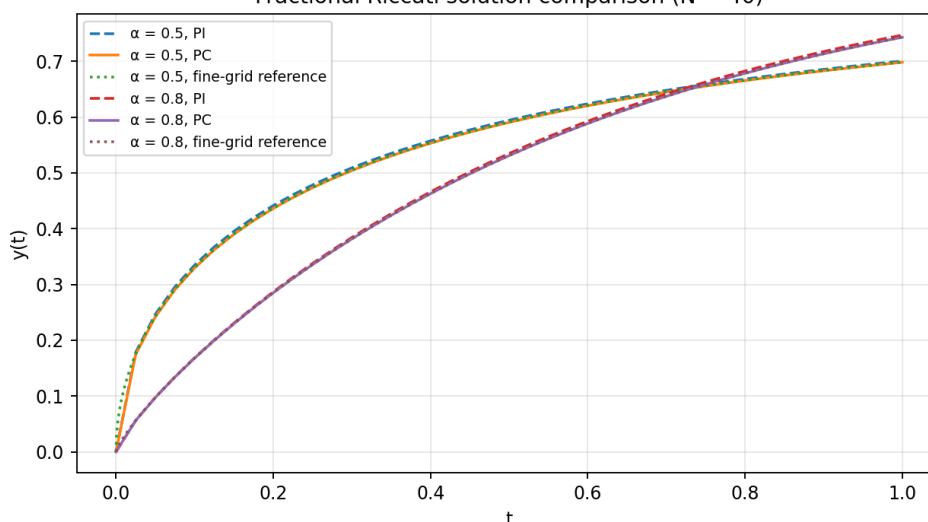


Figure 3. Fractional Riccati solution profiles for two fractional orders, comparing approximations and fine-grid references.

5.4 Riccati empirical convergence

Table 5. Reference-based empirical convergence orders for the Riccati equation

α	Refinement	Product integration	Predictor–corrector
0.5	40 $\rightarrow$ 80	1.07968	1.58132
0.5	80 $\rightarrow$ 160	1.04942	1.56332
0.5	160 $\rightarrow$ 320	1.03161	1.57400
0.8	40 $\rightarrow$ 80	1.01108	1.83937
0.8	80 $\rightarrow$ 160	1.00589	1.82798
0.8	160 $\rightarrow$ 320	1.00310	1.83654

The Riccati estimates broadly support approximately first-order product integration and faster predictor–corrector convergence. Estimates depend on the finite resolution of the numerical reference and should not be regarded as exact asymptotic orders.



6. DISCUSSION

The two benchmarks distinguish analytical verification from numerical approximation. The manufactured problem enables independent comparison against an exact solution, whereas the Riccati equation examines behaviour when only a high-resolution numerical reference is available. In both cases the predictor–corrector method yields smaller errors on tested grids and displays higher empirical convergence orders. These results agree qualitatively with established fractional integration theory but do not prove universal superiority across all nonlinear fractional equations.

Fractional order affects quadrature weights and therefore approximation error. In the manufactured benchmark, the exact solution t^2 is unchanged with α , allowing numerical-order effects to be distinguished from changes to the solution itself. In the Riccati case the actual solution also changes with α . Numerical schemes must account for the history kernel; the error depends on both the quadrature approximation and the regularity of the solution.

7. CONCLUSION

This study compared analytical and numerical techniques for two nonlinear Caputo fractional initial-value problems. The first benchmark provided the exact solution $y(t) = t^2$, while the second used a fine-grid reference solution for a fractional Riccati equation. Under systematic grid refinement, both numerical schemes showed decreasing error. The explicit product-integration method exhibited approximately first-order convergence, while an Adams-type predictor–corrector scheme displayed higher orders broadly consistent with $1 + \alpha$. The predictor–corrector errors were smaller across the investigated cases. These observations support its utility for these problems but should not be generalized to all nonlinear fractional differential equations without additional tests and theory.

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