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Generalized Zygmund Type Inequalities for Polynomials

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ABSTRACT

If $p(z)$is a polynomial of degree n having no zero in $|z|<k,k>0$, then for $0<r\le R\le k$, integerss,$1\le s\le n$ andq >0, we prove

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p^s\left(Re^{i\theta}\right)\right|^q d\theta\right)^{\frac{1}{q}}\le n(n-1).......(n-s+1)D_q\left[\left(\frac{R+k}{r+k}\right)^n\{M(p,r)-m\}\right],$$

Where, $D_q=\left(\frac{1}{2\pi}\int_0^{2\pi}\left|k^s+R^s e^{i\alpha}\right|^q d\alpha\right)^{-\frac{1}{q}}$. ***Our result gives some interesting well-known results as corollaries.***

Keyword: - Polynomial, L^q Inequalities, Maximum Modulus.

1. INTRODUCTION AND STATEMENT OT THE RESULTS

Let $p(z)=\sum_{v=0}^{n}a_v z^v$ be a polynomial of degree n and $p'(z)$ be its derivative, then for $q>0$,

$$\left\{\int_0^{2\pi}\left|p'\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}}\le n\left\{\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}}, \tag{1.1}$$

If we let $q\to\infty$in (1.1) and make use of the well-known fact from analysis [16, 17] that

$$\lim_{r\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}}=\max_{|z|=1}|p(z)|, \tag{1.2}$$

we obtain the following inequalities

$$\max_{|z|=1}|p'(z)|\le n\max_{|z|=1}|p(z)|, \tag{1.3}$$

Inequality (1.3) is a classical result due to Bernstein [4].

If we restrict ourselves to the class of polynomials having no zero in$|z|<1$, then inequality (1.1) can be improved. In fact, the following results are known.

Theorem A. *If $p(z)$ is a polynomial of degree n having no zero in$|z|<1$, then for each$q>0$,*

$$\left\{\int_0^{2\pi}\left|p'\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}}\le nC_q\left\{\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}}, \tag{1.4}$$

Where $C_q=\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|1+e^{i\alpha}\right|^q d\alpha\right\}^{-\frac{1}{q}}$.

In (1.4), *equality occurs for*$p(z)=\alpha z^n+\beta$, $|\alpha|=|\beta|$.

For$q\ge 1$, Theorem A was found by de-Bruijn [6] and later independently proved by Rahman [13]. For the special case$q=2$, it was proved by Lax [12]. Rahman and Schmeisser [14] showed that (1.4) remain valid for $0<q<1$ as well.

For the class of polynomials having no zero in the disc$|z|<k,k\ge 1$, Govil and Rahman [10] proved the following inequality (1.5) for$q\ge 1$.

Later it was shown by Gardner and Weems [9], and independently by Rather [15] that inequality (1.5) also holds for$0<q<1$.

Theorem B. *If $p(z)$ is a polynomial of degree n having no zero in $|z|<k$, $k\ge 1$, then for $q>0$,*

$$\left\{\int_0^{2\pi}\left|p'\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}}\le nF_q\left\{\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^q d\theta\right\}^{\frac{1}{q}}, \tag{1.5}$$

where

$F_q = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|k + e^{i\alpha}\right|^q d\alpha\right\}^{-\frac{1}{q}}.$

Dewan and Bidkham [7] generalized the famous result due to Malik [11] by proving

Theorem C. *If $p(z)$ is a polynomial of degree n such that it has no zero in $|z| < k, k \geq 1$, then for $1 \leq R \leq k$,*

$$\max_{|z|=R}|p'(z)| \leq n\frac{(R+k)^{n-1}}{(1+k)^n}\max_{|z|=1}|p(z)|. \tag{1.6}$$

The result is best possible and extremal polynomial is $p(z) = \left(\frac{z+k}{1+k}\right)^n$.

Barchand and Dewan [5] obtained a generalization as well as an improvement of (1.6) by considering the sth derivative of $p(z)$.

Theorem D. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k, k > 0$, then for $0 < r \leq R \leq k$, and $1 \leq s \leq n$,*

$$\max_{|z|=R}|p^s(z)| \leq \frac{n(n-1)\ldots\ldots(n-s+1)}{R^s+k^s}\left(\frac{R+k}{r+k}\right)^n\left[\max_{|z|=r}|p(z)| - m\right], \tag{1.7}$$

Where $m = \min_{|z|=k}|p(z)|$.

The result is best possible for $s = 1$ and equality in (1.7) holds for $p(z) = (z+k)^n$.

In this paper, we obtain an L^q version of Theorem D which has some interesting consequences. More precisely, we have

Theorem. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k, k > 0$, then for $0 < r \leq R \leq k$, integers $s, 1 \leq s \leq n$ and $q > 0$,*

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p^s\left(Re^{i\theta}\right)\right|^q d\theta\right)^{\frac{1}{q}} \leq n(n-1)\ldots\ldots(n-s+1)C_q\left[\left(\frac{R+k}{r+k}\right)^n\{M(p,r)-m\}\right], \tag{1.8}$$

Where $C_q = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|k^s + R^s e^{i\alpha}\right|^q d\alpha\right)^{-\frac{1}{q}}$ *and* $m = \min_{|z|=k}|p(z)|$.

Remark 1.1. Taking limit as $q \to \infty$ in the inequality (1.8), we obtain inequality (1.7) of Theorem D.

Remark 1.2. If we put $r = s = 1$ in (1.8), we get an integral analogue of a best possible result proved by Aziz and Shah [3, Corollary 5] which is further an improvement of Theorem C due to Dewan and Bidkham [7].

Corollary 1.1. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k, k > 0$, then for $1 \leq R \leq k$, and $q > 0$,*

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p^s\left(Re^{i\theta}\right)\right|^q d\theta\right)^{\frac{1}{q}} \leq nE_q\left[\left(\frac{R+k}{1+k}\right)^n\{M(p,1)-m\}\right],$$

Where $E_q = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|k + Re^{i\alpha}\right|^q d\alpha\right)^{-\frac{1}{q}}$ *and* $m = \min_{|z|=k}|p(z)|$.

Remark 1.3. Further, on putting $R = k = 1$ in our theorem, we have

Corollary 1.2. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for $0 < r \leq 1$, integers $s, 1 \leq s \leq n$ and $q > 0$,*

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p^s\left(e^{i\theta}\right)\right|^q d\theta\right)^{\frac{1}{q}} \leq n(n-1)\ldots\ldots(n-s+1)E_q\left[\left(\frac{2}{r+1}\right)^n\{M(p,r)-m\}\right], \tag{1.9}$$

where $E_q = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|1+e^{i\alpha}\right|^q d\alpha\right)^{-\frac{1}{q}}$ *and* $m = \min_{|z|=k}|p(z)|$.

For $s = r = 1$, corollary 1.2 becomes the L^q version of a result due to Aziz and Dawood [1].

1.1 Lemmas

Lemma 2.1. *If* $p(z)$ *is a polynomial of degree n which does not vanish in* $|z| < k, k \geq 1$, *then for each* $q > 0$ *and integers* $s, 1 \leq s \leq n$,

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p^s\left(e^{i\theta}\right)\right|^q d\theta\right)^{\frac{1}{q}} \leq n(n-1).......(n-s+1)B_q\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p\left(e^{i\theta}\right)\right|^r d\theta\right)^{\frac{1}{q}},$$

Where $B_q = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|k^s + e^{i\alpha}\right|^q d\alpha\right)^{-\frac{1}{q}}$.

This result was proved by Aziz and Shah [2].

Lemma 2.2. *Let* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v, 1 \leq \mu \leq n$, *be a polynomial of degree n having no zero in* $|z| < k, k > 0$, *then for* $0 < r \leq R \leq k$,

$$M(p,R) \leq M(p,r)\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}} - \left\{\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}} - 1\right\}m,$$

where $m = \min_{|z|=k}|p(z)|$.

The result is sharp and equality holds for the polynomial $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ *where n is a multiple of* μ.

This lemma is due to Dewan at. al [8].

1.2 Proof of the Theorem

If $p(z)$ has no zero in $|z| < k$, $k > 0$ and if $0 < r \leq R \leq k$, then $p(Rz)$ has no zero in $|z| < \frac{k}{R}, \frac{k}{R} \geq 1$. For any complex number α such that $|\alpha| < 1$, the polynomial $P(z) = p(Rz) + \alpha m$, where $m = \min_{|z|=k}|p(z)|$ has no zero in $|z| < \frac{k}{R}$. It follows trivially in case m. Suppose $m > 0$, then on the circle $|z| = \frac{k}{R}$, $\min_{|z|=\frac{k}{R}}|p(Rz)| = \min_{|z|=k}|p(z)| = m$ and therefore for $|z| = \frac{k}{R}$, $|\alpha m| < m = |p(Rz)|$. Thus by Rouche's theorem $P(z)$ has no zero in the open disc $|z| < \frac{k}{R}$. If we apply Lemma 2.1 to $P(z)$, we get

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|P^s\left(e^{i\theta}\right)\right|^q d\theta\right)^{\frac{1}{q}} \leq n(n-1).......(n-s+1)C_q\left(\frac{1}{2\pi}\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^r d\theta\right)^{\frac{1}{q}}, \quad (3.1)$$

Where $C_q = \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\left(\frac{k}{R}\right)^s + e^{i\alpha}\right|^q d\alpha\right)^{-\frac{1}{q}}$.

Inequality (3.1) is equivalent to

$$R^s\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p^s\left(Re^{i\theta}\right)\right|^q d\theta\right)^{\frac{1}{q}} \leq n(n-1).......(n-s+1)B_q\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p\left(Re^{i\theta}\right)+\alpha m\right|^r d\theta\right)^{\frac{1}{q}},$$

That is,

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p^s\left(Re^{i\theta}\right)\right|^q d\theta\right)^{\frac{1}{q}} \leq n(n-1).......(n-s+1)D_q\left(\frac{1}{2\pi}\int_0^{2\pi}\left|p\left(Re^{i\theta}\right)+\alpha m\right|^r d\theta\right)^{\frac{1}{q}}, ...(3.2)$$

Where $D_q = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| k^s + R^s e^{i\alpha} \right|^q d\alpha \right)^{-\frac{1}{q}}$

Now, we have for any θ with $0 \le \theta < 2\pi$,

$$|p(Re^{i\theta}) + \alpha m| \le \max_{|z|=1} |p(Rz) + \alpha m|. \tag{3.3}$$

Suppose at some z_0 on $|z| = 1$, $|p(Rz) + \alpha m|$attains its maximum.

Then $\max_{|z|=1} |p(Rz) + \alpha m| = |p(Rz_0) + \alpha m|$.

In $|p(Rz_0) + \alpha m|$, we choose suitable argument of αsuch that

$$|p(Rz_0) + \alpha m| = |p(Rz_0)| - |\alpha| m$$

$$\le \max_{|z|=1} |p(Rz)| - |\alpha| m. \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$|p(Re^{i\theta}) + \alpha m| \le \max_{|z|=1} |p(Rz)| - |\alpha| m. \tag{3.5}$$

Using Lemma 2.2 for $\mu = 1$ in (3.5), we have

$$|p(Re^{i\theta}) + \alpha m| \le M(p,r) \left(\frac{R+k}{r+k}\right)^n - \left\{\left(\frac{R+k}{r+k}\right)^n - 1\right\} m - |\alpha| m. \tag{3.6}$$

If we make use of inequality (3.6) in (3.2), we are lead to

$$\left(\frac{1}{2\pi} \int_0^{2\pi} \left| p^s\left(Re^{i\theta}\right) \right|^q d\theta \right)^{\frac{1}{q}} \le n(n-1).......(n-s+1) D_q$$

$$\times \left[M(p,r) \left(\frac{R+k}{r+k}\right)^n - \left\{ \left(\frac{R+k}{r+k}\right)^n - 1 \right\} m - |\alpha| m \right] \tag{3.7}$$

Finally by letting $|\alpha| \to 1$in (3.7), we obtain the desired result and the proof of the theorem is completed.

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