Sharpening of a Polynomial Inequality of T.J. Rivlin

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ABSTRACT

If $p(z)$ is a polynomial of degree n having no zero in $|z|<1$, then for $r\le 1$, the famous result proved by T.J. Rivlin, [Amer, Math. Monthly, 67(1960), 251-253] is

$$M(p,r)\ge\left(\frac{1+r}{2}\right)^n M(p,1).$$

This inequality is sharp. In this paper, by involving some coefficients of the polynomial, we improve as well as generalize this inequality and has interesting consequences.
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1. INTRODUCTION AND STATEMENT OT THE RESULTS

Let $p(z)=\sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ be a polynomial of degree n, and let $M(p,r)=\max_{|z|=r}|p(z)|$. Then the following inequalities concerning the maximum modulus of a polynomial on a circle $|z|=R$ are known.

$$M(p,R)\le R^n M(p,1), R\ge 1. \tag{1.1}$$

and

$$M(p,r)\ge r^n M(p,1), r\le 1 \tag{1.2}$$

Inequalities (1.1) *and* (1.2) *are sharp and equality holds for $p(z)=\lambda z^n$, $\lambda\neq 0$ being a complex number.*

Inequality (1.1) is a simple deduction from the Maximum Modulus Principle [5, 3]. Inequality (1.2) is due to Zarantonello and Varga [7].

If we restrict ourselves to the class of polynomials not vanishing in $|z|<1$, inequality analogous to (1.2) was obtained by Rivlin [6].

Theorem A. *If $p(z)$ is a polynomial of degree n having no zero in $|z|<1$, then for $r\le 1$,*

$$M(p,r)\ge\left(\frac{1+r}{2}\right)^n M(p,1). \tag{1.3}$$

Inequality (1.3) *is sharp and equality holds for the polynomial $p(z)=\left(\frac{\alpha+\beta z}{2}\right)^n$, where $|\alpha|=|\beta|$.*

Govil generalized Theorem A [1, Theorem 1] by proving.

Theorem B. *If $p(z)=\sum_{\upsilon=0}^{n} a_\upsilon z^\upsilon$ is a polynomial of degree n having no zero in $|z|<1$, then for $0<r\le R\le 1$,*

$$M(p,r)\ge\left(\frac{1+r}{1+R}\right)^n M(p,R). \tag{1.4}$$

The result is best possible and equality holds for the polynomial $p(z)=(z+1)^n$.

Recently, Govil and NwaeZe [2] have considered a more general class of polynomials $p(z)=a_0+\sum_{\upsilon=\mu}^{n} a_\upsilon z^\upsilon$, $1\le\mu<n$ not vanishing in $|z|<1$ and proved an extension as well as sharpening of Rivlin's inequality (1.3).

Theorem C. *Let $p(z)=a_0+\sum_{\upsilon=\mu}^{n} a_\upsilon z^\upsilon$ is a polynomial of degree n having no zero in $|z|<1$, then for $0<r<1$,*

$$M(p,r) \geq \frac{\left(1+r^{\mu}\right)^{\frac{n}{\mu}}}{\left(1+r^{\mu}\right)^{\frac{n}{\mu}} + \mu\left\{(2)^{\frac{n}{\mu}} - (1+r)^{\frac{n}{\mu}}\right\}} \left\{M(p,1) + n\, m In\left(\frac{2}{1+r}\right)\right\}. \tag{1.7}$$

where
$m = \min_{|z|=1} |p(z)|$.

In this paper, under the same set of assumptions of Theorem C, by involving certain coefficients, we prove the following inequality which improves the bound given by Theorem C. More precisely, we obtain

Theorem. *Let* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$ *be a polynomial of degree n having no zero in* $|z| < 1, k \geq 1$, *then for* $0 < r \leq 1$,

$$M(p,r) \geq \left\{M(p,1) + mnIn\left(\frac{2}{1+r}\right)\right\} \frac{1}{1+n\left[\int_r^1 \frac{1}{1+t} exp\left\{n \int_r^t \frac{x^{\mu} + \frac{\mu|a_{\mu}|}{n|a_0|} x^{\mu-1}}{x^{\mu+1} + \frac{\mu|a_{\mu}|}{n|a_0|}(x^{\mu}+t)+1} dx\right\} dt\right]}, \tag{1.8}$$

Where $m = \min_{|z|=1} |p(z)|$.

Remark 1.1. By lemma 2.5, it is evident that the R.H.S (1.8) dominates over that of (1.7) and thus our theorem gives better bound.

Remark 1.2. Putting $\mu = 1$in Theorem C, we have under the same hypotheses, the following improvement of the famous result due to Rivlin [6].

Corollary 1.1. *If* $p(z)$ *is a polynomial of degree n having no zero in* $|z| < 1$, *then for* $0 < r \leq 1$,

$$M(p,r) \geq \left(\frac{1+r}{2}\right)^n \left\{M(p,1) + nmIn\left(\frac{2}{1+r}\right)\right\} \tag{1.9}$$

Where $m = \min_{|z|=1} |p(z)|$.

Inequality (1.9) *is sharp and equality holds for the polynomial* $p(z) = \left(\frac{\alpha+\beta z}{2}\right)^n$, *where* $|\alpha| = |\beta|$.

Remark 1.2. If we put $\boldsymbol{\mu = 1}$ in Lemma 2.5, we have, in particular

$$.\left(\frac{2}{1+r}\right)^n - 1 \geq \frac{1}{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|}r+r^2\right)^{\frac{n}{2}}} \int_r^1 \frac{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|}t+t^2\right)^{\frac{n}{2}}}{1+t} dt. \tag{1.10}$$

Further, if we put $\mu = 1$in our theorem and make use of inequality (1.10), we have

Corollary 1.2. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree n having no zero in* $|z| < 1$, *then for* $0 < r \leq 1$,

$$M(p,r) \geq \left\{M(p,1) + mnIn\left(\frac{2}{1+r}\right)\right\} \frac{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|}r+r^2\right)^{\frac{n}{2}}}{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|}r+r^2\right)^{\frac{n}{2}} + n\int_r^1 \frac{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|}t+t^2\right)^{\frac{n}{2}}}{1+t}dt}. \quad (1.11)$$

Inequality (1.11) *is sharp and equality holds for the polynomial* $p(z) = \left(\frac{\alpha+\beta z}{2}\right)^n$, *where* $|\alpha| = |\beta|$.

Again, the quantity $\left\{\frac{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|}r+r^2\right)^{\frac{n}{2}}}{1+n\int_r^1 \frac{\left(1+\frac{2}{n}\frac{|a_1|}{|a_0|}t+t^2\right)^{\frac{n}{2}}}{1+t}dt}\right\}$ appearing in the R.H.S of (1.11) is greater than or equal to $\left(\frac{1+r}{2}\right)^n$ and hence corollary 1.2 further improves corollary 1.1, which, in turn, improves Theorem C due to Rivlin [6].

1.1 Lemmas

To prove the theorem, the following lemmas are required.

Lemma 2.1. *If* $p(z)$ *, is a polynomial of degree n having no zero in the disc* $|z| < k, k \geq 1$, *then*

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k}\left\{\max_{|z|=1}|p(z)| - \min_{|z|=k}|p(z)|\right\}. \quad (2.1)$$

Lemma 2.2. *If* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, *is a polynomial of degree n not vanishing in* $|z| < 1$, *then for* $0 < r \leq R \leq 1$,

$$M(p,r) \geq exp\left\{-n\int_r^R \frac{t^\mu + \frac{\mu|a_\mu|}{n|a_0|}t^{\mu-1}}{t^{\mu+1} + \frac{\mu|a_\mu|}{n|a_0|}(t^\mu+t)+1}dt\right\} M(p,R). \quad (2.2)$$

and

$$\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^\mu \leq 1. \quad (2.3)$$

This lemma was proved by Qazi [4, Theorem 1 and Remark 1].

Lemma 2.4. *If* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v, 1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zero in* $|z| < 1$, *then for* $0 < r \leq R \leq 1$,

$$\boldsymbol{exp}\left\{-\boldsymbol{n}\int_r^R \frac{\frac{\mu|a_\mu|}{n|a_0|}t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu|a_\mu|}{n|a_0|}(t^\mu+t)+1}\boldsymbol{dt}\right\} \geq \left(\frac{1+r^\mu}{1+R^\mu}\right)^{\frac{n}{\mu}}. \quad (2.4)$$

Proof of Lemma 2.4. Since $p(z) \neq 0$ in$|z| < 1$, the polynomial $P(z) = p(tz) \neq 0$in $|z| < \frac{1}{t}$, $\frac{1}{t} \geq 1$ where $0 < t \leq 1$ Hence applying inequality (2.3) of Lemma 2.2 to $P(z)$, we get

$$\frac{|a_\mu| t^\mu}{|a_0|} \left(\frac{1}{t}\right)^\mu \leq \frac{n}{\mu}, \tag{2.5}$$

where $m = \min_{|z|=\frac{1}{t}} |p(z)| = \min_{|z|=\frac{1}{t}} |p(tz)| = \min_{|z|=1} |p(z)|$.

Now, (2.5) becomes

$$\frac{|a_\mu|}{|a_0|} \leq \frac{n}{\mu},$$

which is equivalent to

$$\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(t^\mu + t) + 1} \leq \frac{t^{\mu-1}}{t^\mu + 1}. \tag{2.6}$$

Integrating both sides of (2.6) with respect to t from r to R where $0 < r \leq R \leq 1$, we have

$$\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(t^\mu + t) + 1} dt \leq \int_r^R \frac{t^{\mu-1}}{t^\mu + 1} dt,$$

which is equivalent to

$$-n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}(t^\mu + t) + 1} dt \leq -n\int_r^R \frac{t^{\mu-1}}{t^\mu + 1} dt,$$

which implies

$$\exp\left\{-n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}\left(t^\mu + t\right) + 1} dt\right\} \leq \exp\left(-n\int_r^R \frac{t^{\mu-1}}{t^\mu + 1} dt\right) = \left(\frac{1+r^\mu}{1+R^\mu}\right)^{\frac{n}{\mu}}.$$

which proves Lemma 2.4 completely.

Lemma 2.5. *If* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v, 1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zero in* $|z| < 1$, *then for* $0 < r \leq 1$,

$$\boldsymbol{\mu \frac{1}{(1+r^\mu)^{\frac{n}{\mu}}}\left\{(2)^{\frac{n}{\mu}} - (1+r)^{\frac{n}{\mu}}\right\} \geq n\int_r^1 \frac{1}{1+t} exp\left\{n\int_r^t \frac{\frac{\mu|a_\mu|}{n|a_0|} x^{\mu-1} + x^\mu}{x^{\mu+1} + \frac{\mu|a_\mu|}{n|a_0|}(x^\mu + x) + 1} dx\right\} dt.} \tag{2.7}$$

Proof of Lemma 2.5. Since$\boldsymbol{0 < r \leq t \leq 1}$**,** on applying Lemma 2.5, we have

$$\int_r^1 \frac{n}{1+t}\left(\frac{1+t^\mu}{1+r^\mu}\right)^{\frac{n}{\mu}} dt \geq \int_r^1 \frac{n}{1+t} exp\left\{n\int_r^t \frac{\frac{\mu|a_\mu|}{n|a_0|} x^{\mu-1} + x^\mu}{x^{\mu+1} + \frac{\mu|a_\mu|}{n|a_0|}(x^\mu + x) + 1} dx\right\} dt. \tag{2.8}$$

As $0 < t \leq 1$, $1 + t^\mu \leq 1 + t$ for $\mu = 1,2, \ldots\ldots\ldots, n$.

$$\int_r^1 \frac{n}{1+t}\left(\frac{1+t^\mu}{1+r^\mu}\right)^{\frac{n}{\mu}} dt \le \int_r^1 \frac{n}{1+t}\left(\frac{1+t}{1+r^\mu}\right)^{\frac{n}{\mu}} dt$$

$$= \mu \frac{n}{\mu} \frac{1}{(1+r^\mu)^{\frac{n}{\mu}}} \int_r^1 (1+t)^{\frac{n}{\mu}-1} dt$$

$$= \mu \frac{1}{(1+r^\mu)^{\frac{n}{\mu}}} \left[(2)^{\frac{n}{\mu}} - (1+r)^{\frac{n}{\mu}}\right].$$

Or $$\int_r^1 \frac{n}{1+t}\left(\frac{1+t^\mu}{1+r^\mu}\right)^{\frac{n}{\mu}} dt \le \mu \frac{1}{(1+r^\mu)^{\frac{n}{\mu}}} \left[(2)^{\frac{n}{\mu}} - (1+r)^{\frac{n}{\mu}}\right]. \tag{2.9}$$

Combining (2.8) and (2.9), we obtain the required result.

1.2 Proof of the Theorem

Since $p(z)$ has no zero in $|z| < 1$, then for $0 < t \le 1$, the polynomial $P(z) = p(tz)$ has no zero in $|z| < \frac{1}{t}$, where $\frac{1}{t} \ge 1$. Hence applying Lemma 2.1 to the polynomial $P(z)$, we get

$$\max_{|z|=1} |P'(z)| \le \frac{n}{\left\{1+\left(\frac{1}{t}\right)\right\}} \left\{\max_{|z|=1} |P(z)| - \min_{|z|=\frac{1}{t}} |P(z)|\right\}$$

which implies

$$\max_{|z|=t} |p'(z)| \le \frac{1}{1+t} \left\{\max_{|z|=t} |p(z)| - \min_{|z|=1} |p(z)|\right\}. \tag{3.1}$$

Now, for $0 < r \le 1$ and $0 \le \theta \le 2\pi$, we have $\left|p(e^{i\theta}) - p(re^{i\theta})\right| \le \int_r^1 \left|p'(te^{i\theta})\right| dt$

and it is implied that $\left|p(e^{i\theta})\right| \le \left|p(re^{i\theta})\right| + \int_r^1 \left|p'(te^{i\theta})\right| dt,$

which on using (3.1) leads to $\left|p(e^{i\theta})\right| \le \left|p(re^{i\theta})\right| + \left\{\int_r^1 \frac{n}{1+t} M(p,t)dt - \int_r^1 \frac{n}{1+t} m(p,1)dt\right\}.$

which implies on considering maximum over θ that

$$M(p,1) \le M(p,r) + \left\{\int_r^1 \frac{n}{1+t} M(p,t)dt - \int_r^1 \frac{n}{1+t} m(p,1)dt\right\}. \tag{3.2}$$

Since $r \le t$, by applying inequality (2.2) of Lemma 2.2 with $R = t$, we have

$$M(p,t) \le exp\left\{n \int_r^t \frac{x^\mu + \frac{\mu|a_\mu|}{n|a_0|} x^{\mu-1}}{x^{\mu+1} + \frac{\mu|a_\mu|}{n|a_0|}(x^\mu + t) + 1} dx\right\} M(p,r). \tag{3.3}$$

Using (3.3) to (3.2), we obtain

$$M(p,1) \le M(p,r) + nM(p,r)\left[\int_r^1 \frac{1}{1+t} exp\left\{n \int_r^t \frac{x^\mu + \frac{\mu|a_\mu|}{n|a_0|} x^{\mu-1}}{x^{\mu+1} + \frac{\mu|a_\mu|}{n|a_0|}(x^\mu + t) + 1} dx\right\} dt\right] - \int_r^1 \frac{n}{1+t} m(p,1)dt$$

$$= M(p,r) + nM(p,r)\left[\int_r^1 \frac{1}{1+t} exp\left\{n \int_r^t \frac{x^\mu + \frac{\mu|a_\mu|}{n|a_0|} x^{\mu-1}}{x^{\mu+1} + \frac{\mu|a_\mu|}{n|a_0|}(x^\mu + t) + 1} dx\right\} dt\right] - m(p,1) n In\left(\frac{2}{1+r}\right). \tag{3.4}$$

Inequality (3.4) is equivalent to

$$M(p,r) \ge \left\{M(p,1) + m(p,1) n In\left(\frac{2}{1+r}\right)\right\} \frac{1}{1 + n\left[\int_r^1 \frac{1}{1+t} exp\left\{n \int_r^t \frac{x^\mu + \frac{\mu|a_\mu|}{n|a_0|} x^{\mu-1}}{x^{\mu+1} + \frac{\mu|a_\mu|}{n|a_0|}(x^\mu + t) + 1} dx\right\} dt\right]}.$$

and hence completes the proof of the Theorem.

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